Exam 2,

13.4: Acceleration/Velocity/Dist

14.1/3/4/7: **Partial derivatives**

Level Curves, Domain, Partials, Tangent Plane, local max/min (2nd deriv. test), global max/min, applied max/min

15.1-15.3: **Double integrals**

general regions (top/bot, left/right), reversing order, polar.

Section 15.4 (center of mass) will NOT be on the test.

<u>How to do ALL problems in ch. 15</u>

Step 1: Find integrand(s). Solve for the z's.

Step 2: Draw the region

2(a) Draw given xy-bounds.

2(b) Draw intersection of z's.

Step 3: Set-up bounds.

Step 4: Evaluate

Three ways to set-up (step 3)

"Bottom/Top":

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

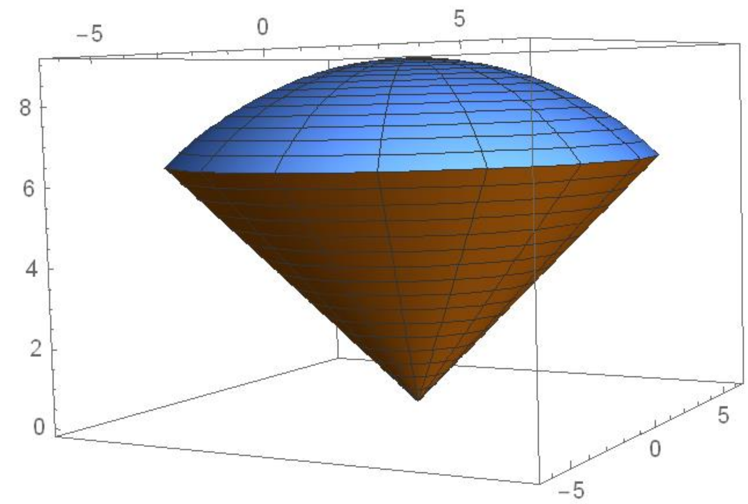
"Left/Right":

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

"Inside/Outside": Polar (NEW, today)

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

***Entry Task*:** How do you start this?



HW 15.3: Find the volume above the upper cone $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 81$

15.3 Double Integrals over Polar Regions

Recall:

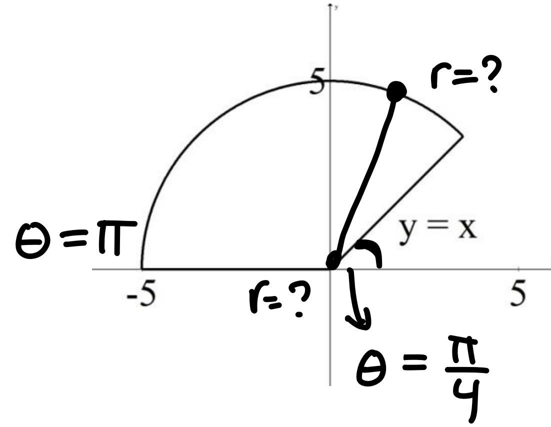
- θ = angle measured from positive x-axis
- r = distance from origin
- $x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2 + y^2 = r^2$

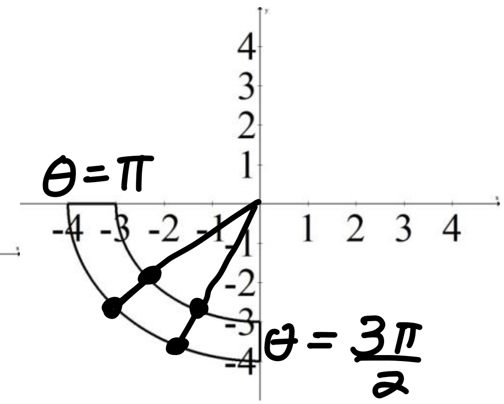
To set up a double integral in polar we:

1. Describing the region in polar
2. Replace "x" by "r cos(θ)"
3. Replace "y" by "r sin(θ)"
4. Replace "dA" by "r dr dθ"

Step 1: Describing regions in polar.

Examples: Describe the regions





$$\int_{\frac{\pi}{4}}^{\pi}\left(\int_{0}^{5} f(x,y)\, r\,dr\right)d\theta$$

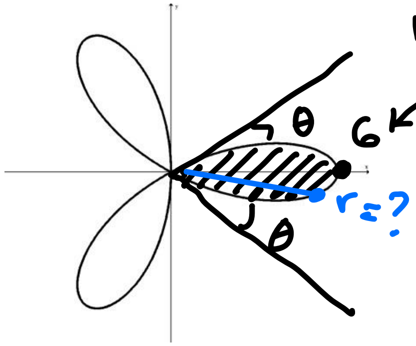
$$\int_{\pi}^{\frac{3\pi}{2}}\left(\int_{3}^{4} f(x,y)\, r\,dr\right)d\theta$$

in direction of θ, so positive

HW 15.3: Describe the region of one $\boxed{\text{loop}}$ of r = 6cos(3θ).

$r=0$

$r = 6\cos(0) = 6$



$r=0 \Rightarrow 6\cos(3\theta)=0$

$\cos(\text{blah})=0$

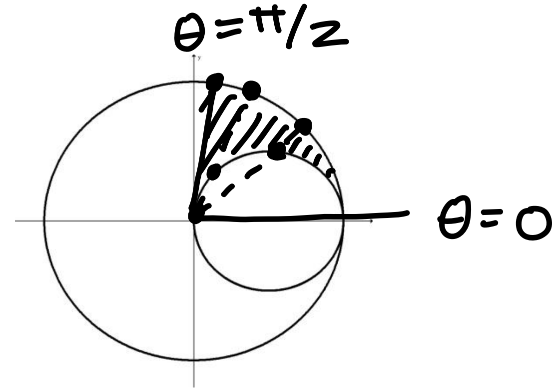
$\text{blah} = \ldots, -\pi/2, \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\theta = \ldots, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ divide by 3

Fit the graph, both true consecutivley

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{0}^{6\cos(3\theta)} f(x,y)\, r\,dr\,d\theta$$

HW 15.3: Describe the region in the first quadrant between the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 4x$.



$$\int_{0}^{\pi/2} \int_{4\cos(\theta)}^{4} f(x,y)\, r\,dr\,d\theta$$

outside $\Rightarrow x^2+y^2=16$

$x^2+y^2=r^2$, $r=4$

inside $\Rightarrow x^2+y^2=4x$

$x^2+y^2=r^2$ $\quad x=r\cos(\theta)$

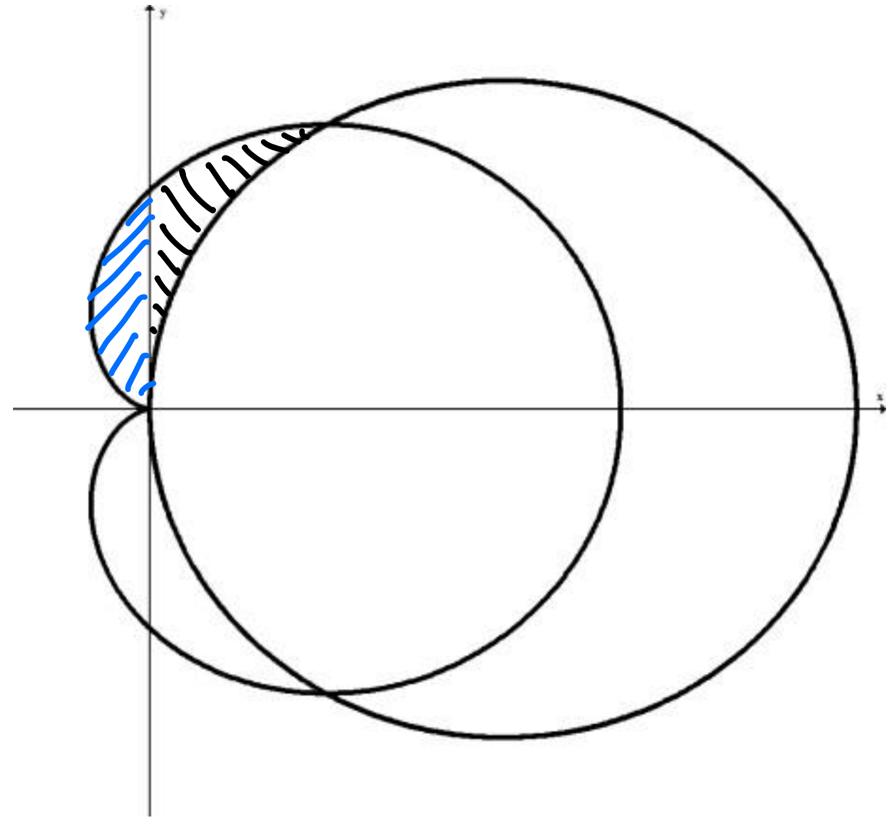
$$\frac{r^2 = 4r\cos(\theta)}{r}$$

$r = 4\cos(\theta)$

HW 15.3/7:

Describe the region inside $r = 1 + \cos(\theta)$ and outside $r = 3\cos(\theta)$.

do both regions
seprately



Visuals: https://www.desmos.com/calculator/wc62p5zmy9

Step 2: Set up your integral in polar.

$x = r\cos\theta \quad y = r\sin\theta$

Conceptual notes:

$V = f(x,y)\,dA$

Cartesian

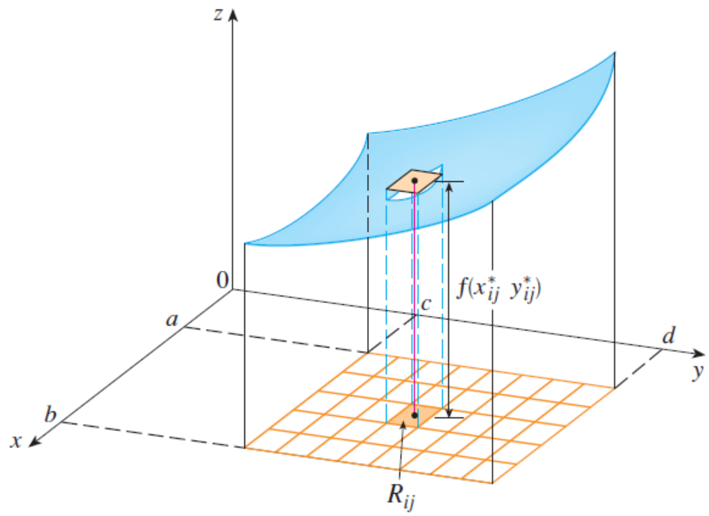
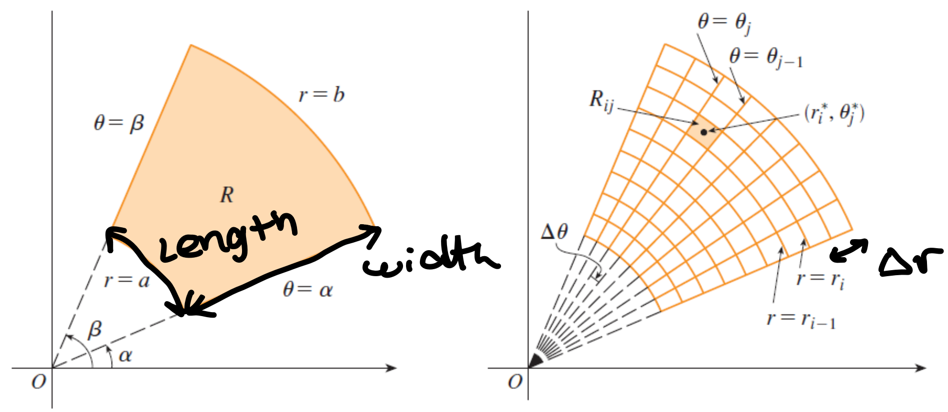
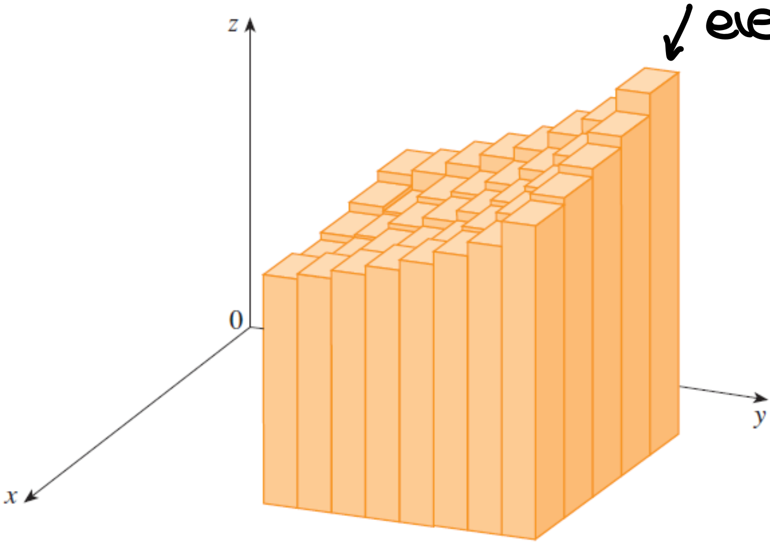


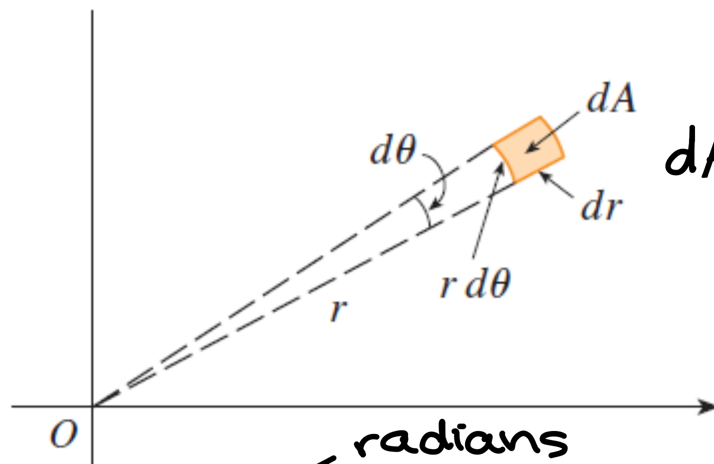
FIGURE 4

Polar



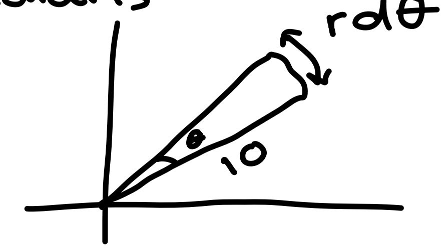
volume element





$dA = r\,dr\,d\theta$

distance $= r\theta$ ← radians



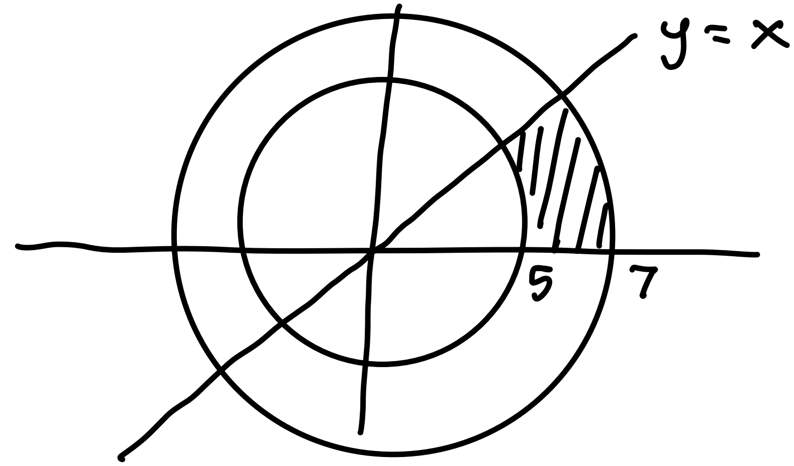
$\Delta A = \Delta x \Delta y = \Delta y \Delta x = dx\,dy = dy\,dx$

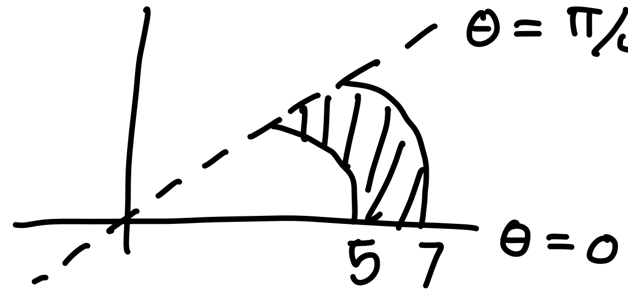
Examples:

1. Compute

$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\,dA$$

where R is the region in the first quadrant that is between $x^2 + y^2 = 49$, $x^2 + y^2 = 25$ and below $y = x$.





$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$

$$\int_0^{\pi/4}\int_5^7 \frac{\cos(r)}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$$

$$\int_0^{\pi/4} \sin(r)\Big|_5^7\,d\theta$$

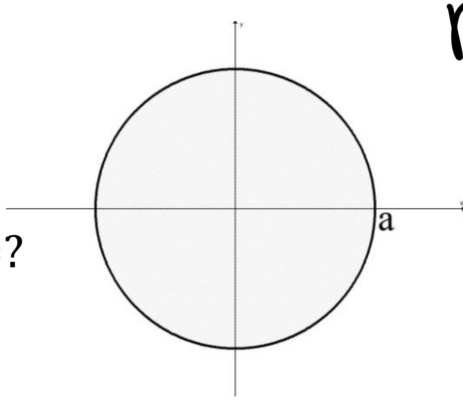
$$\int_0^{\pi/4} \sin(7)-\sin(5)\,d\theta$$

$$(\sin(7)-\sin(5))\theta\Big|_0^{\pi/4} = \boxed{\frac{\pi}{4}\left[\sin(7)-\sin(5)\right]}$$

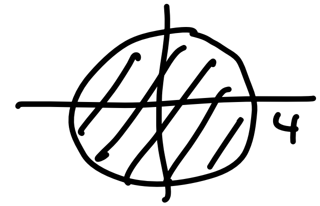
2. Set up the two double integrals below over the entire circular disc of radius a:

$$\iint_D 1\, dA = ?$$

$$\iint_D \sqrt{a^2 - x^2 - y^2}\, dA = ?$$



Find the area of polar region



$$\iint_D 1 \quad dA$$

$$\int_0^{2\pi} \int_0^{4} 1 \quad r\,dr\,d\theta$$

$$\frac{1}{2} r^2 \Big|_0^4 = 8$$

$$\int_0^{2\pi} 8 \; d\theta = \boxed{16\pi}$$

area of Circle! (πr^2)

$$\iint_D \sqrt{a^2 - x^2 - y^2}\; dA$$

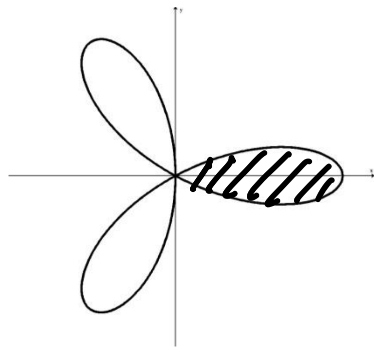
$$\int_0^{2\pi} \int_0^{4} \sqrt{16 - r^2} \quad r\,dr\,d\theta$$

$$\boxed{\frac{2}{3}\pi (4)^3}$$

← (half) volume of sphere! $(\frac{4}{3}\pi r^3)$

3. **HW 15.3**:
Find the area of one closed loop of
r = 6cos(3θ).

$$\iint_D 1 \; r\,dr\,d\theta$$



$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{6\cos(3\theta)} 1 \; r\,dr\,d\theta$$

Solve:

double
angle
property!

4. **HW 15.3**:
Evaluate

$$\iint_R x\, dA$$

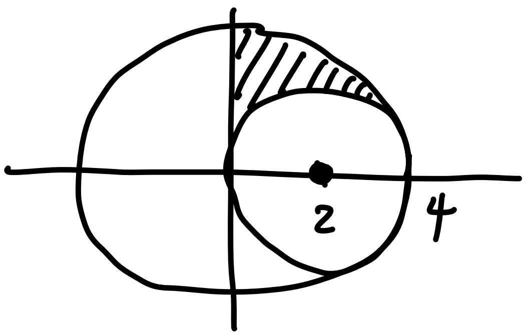
over the region in the first quadrant between the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 4x$ using polar.

CTS!

$x^2 - 4x + y = 0$

$(x-2)^2 + y^2 = 4$

$$\int_0^{\pi/2} \int_{4\cos\theta}^{4} r\cos\theta \; r\, dr\, d\theta$$



$$\int_0^{\pi/2} \cos\theta \left(\frac{1}{3} r^3 \Big|_{4\cos\theta}^{4} \right) d\theta$$

EVEN = half angles

$$\frac{64}{3} \int_0^{\pi/2} \cos\theta - \cos^4\theta \, d\theta$$

$x^2+y^2=16$

$r^2=16$

$r=\pm 4$

$x^2+y^2=4x$

$r^2 = 4r\cos\theta$

$r=4\cos\theta$

$$\frac{64}{3}\int_0^{\pi/2} \cos\theta\, d\theta - \frac{64}{3}\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\frac{1}{2}(1+\cos 2\theta) d\theta$$

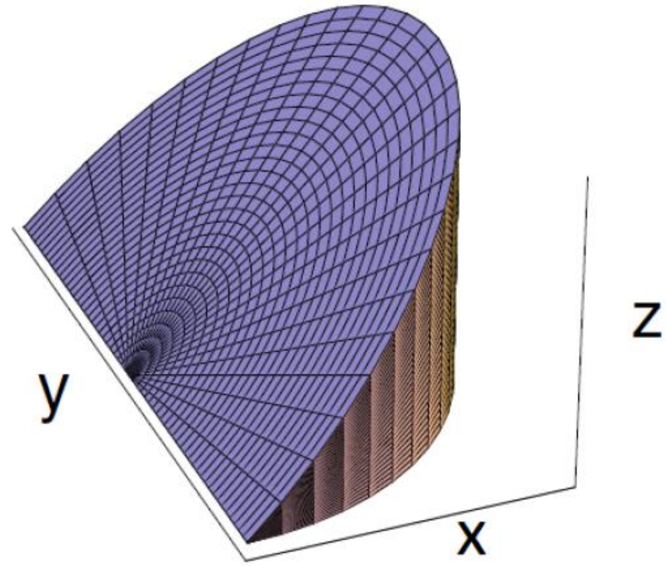
$$\frac{64}{3}\sin\theta\Big]_0^{\pi/2} - \frac{16}{3}\int_0^{\pi/2} \frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta) d\theta$$

$$\frac{64}{3} - \frac{16}{3}\left[\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\Big|_0^{\pi/2}\right]$$

More examples…

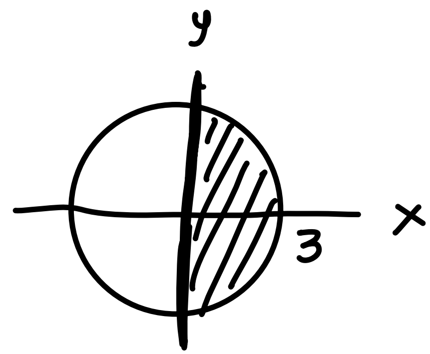
Old Final Exam Problem

Find the volume of the wedge shaped solid that lies above the xy-plane, below the plane $z = x$ and within the solid cylinder $x^2 + y^2 \le 9$.



$$\iint_R x \, dA$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r\cos\theta \; r\,dr\,d\theta$$



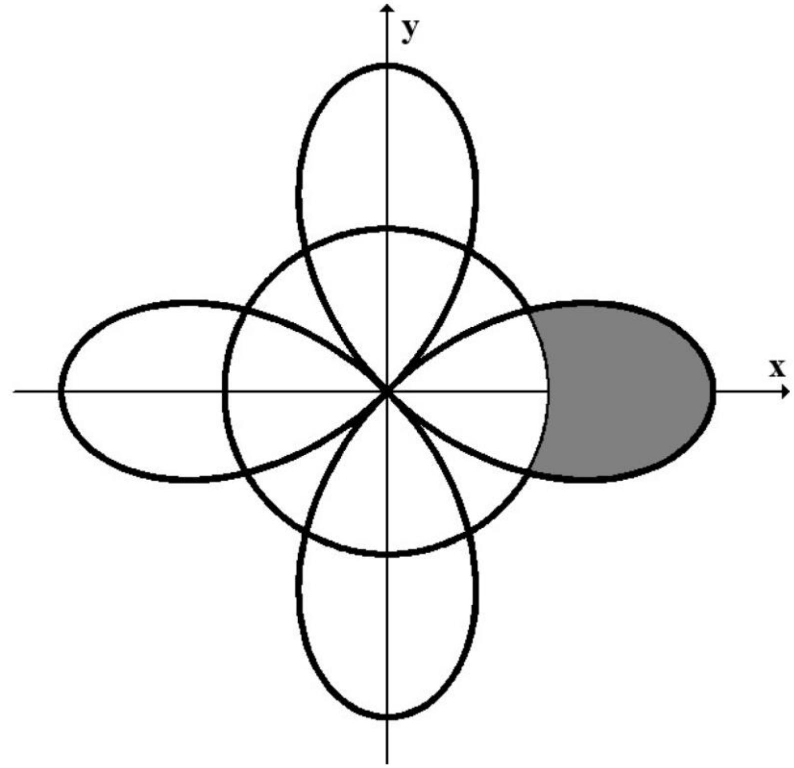
$z = x$ intersect

$z = 0$ @ $x = 0$

Visual: https://www.math3d.org/tPJRHjh1

F13 – Exam 2 – Loveless

Find the area of the region outside the circle $x^2 + y^2 = \frac{1}{4}$ and inside the rose $r = \cos(2\theta)$.



Visual: https://www.desmos.com/calculator/ychsxgwssd

W’16 – Exam 2 – Loveless

Let R be the region in the first quadrant inside the circle $x^2 + y^2 = 9$, and outside the circle $x^2 + y^2 = 2x$. Evaluate

$$\iint_R \frac{y}{x^2 + y^2}\, dA$$

Visual of region: https://www.desmos.com/calculator/4d6alpxm62
Visual of solid: https://www.math3d.org/R7t5OIoA